

Multiplicative Propagation of Error During Recursive Wavelet Estimation

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Abstract

Wavelet coefficients are estimated recursively at progressively coarser scales recursively. As a result, the estimation is prone to multiplicative propagation of truncation errors due to quantization and round-off at each stage. Yet, the influence of this propagation on wavelet filter output has not been explored systematically. Through numerical error analysis of a simple, generic sub-band coding scheme with a half-band low pass finite impulse-response filter for down sampling, we show that truncation error in estimated wavelet filter coefficients can quickly reach unacceptable levels, and may render the results unreliable especially at coarser scales.

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INTRODUCTION

With the exception of Haar wavelets, wavelet coefficients (e.g., in Coiflets or Daubechies wavelets) are approximations to equations without closed-form solutions. Coefficients are recursively estimated at progressively coarser scales, using the estimate on one scale as the input to the next. Such recursive schemes are prone to multiplicative propagation of errors due to quantization

and round-off ("truncation errors") in estimation in each scale. Sub-band coding (i.e., breaking the signal into a number of different frequency bands and encoding each one independently) and downsampling at each scale may determine the pattern of errors. Moreover, these truncation errors are not improved by sample size, and can dominate the variance of estimation. Yet, the propagation of truncation errors and their influence on estimated wavelet coefficients have not been ex-

plored systematically. In this note, we report the results from an error analysis of a simple, generic sub-band coding scheme with a half-band low pass finite impulse-response (FIR) filter for down sampling. We demonstrate the sub-band coding scheme, and derive the bounds for signal-to-noise ratio of the filter output in Section , and provide a numerical analysis of truncation error propagation in Section .

SIGNAL-TO-NOISE RATIO, SUB-BAND CODING, AND DOWNSAMPLING

Due to the dependencies between the variables and the complexity of the expression, an exact formula for the signal-to-noise ratio (SNR) is intractable in general. However, an approximate expression (in decibels) can be given for the SNR

$$10 \log_{10} \mathbb{E} \left[\frac{\mathcal{S}}{\mathcal{N}} \right] \quad (1)$$

where \mathcal{S} and \mathcal{N} denote the power of the signal and noise. With small relative error, the expression in brackets can be replaced with $\frac{\bar{\mathcal{S}}}{\bar{\mathcal{N}}}$, where $\bar{\cdot}$ denotes the mean. Assuming the respective coefficients of variation, $\mathbf{c}_{\mathcal{S}} = \frac{\sigma(\mathcal{S})}{\bar{\mathcal{S}}}$, and $\mathbf{c}_{\mathcal{N}} = \frac{\sigma(\mathcal{N})}{\bar{\mathcal{N}}}$ are small (i.e. bounded by a small constant ϵ), the minimal relative error can be obtained as

$$\frac{\mathcal{S}}{\mathcal{N}} - \frac{\bar{\mathcal{S}}}{\bar{\mathcal{N}}} = \frac{\bar{\mathcal{S}}}{\bar{\mathcal{N}}} \left(\frac{\left[\frac{\mathcal{S} - \bar{\mathcal{S}}}{\sigma_{\mathcal{S}}} \right] \mathbf{c}_{\mathcal{S}} - \left[\frac{\mathcal{N} - \bar{\mathcal{N}}}{\sigma_{\mathcal{N}}} \right] \mathbf{c}_{\mathcal{N}}}{1 + \left[\frac{\mathcal{N} - \bar{\mathcal{N}}}{\sigma_{\mathcal{N}}} \right] \mathbf{c}_{\mathcal{N}}} \right) \quad (2)$$

after a simple algebraic manipulation. The term in parenthesis consists of two approximately normally distributed random variables (given a large sample size) with zero mean and a small variance, divided by a normally distributed random variable with mean one and a small variance. Therefore, the difference on the left-hand side will be relatively small with high probability. From now on, we assume the numerator and denominator in equation 1 can be replaced by its expectation.

Let f_k be the downsampling filter. Then, for a fixed input S using a noise free filter of length N , the detail coefficients d_{yz} take the form

$$d_{yz} = S_{2^z y} - \sum_{i_k} \prod_{\substack{k=1 \\ i_k \in \{1:N\}}}^z f_{i_k} S_{2^z y - \sum_{k=1}^z 2^{k-1} i_k} \quad (3)$$

When each of the filter coefficients f_i is perturbed by noise ($\epsilon_i \sim \mathcal{N}(0, \lambda_i)$), where λ_i is chosen to fix the input signal-to-noise ratio, SNR_I , the perturbed detail coefficient, \tilde{d}_{yz} , takes the form

At scale level z , the output signal to noise ratio (SNR_O) is then given (in decibels) by

$$\tilde{d}_{yz} = d_{yz} = S_{2^z y} - \prod_{k=1}^z (f_{i_k} + \epsilon_{i_k}) S_{2^z y - \sum_{k=1}^z 2^{k-1} i_k} \quad 10 \log_{10} \frac{\mathbb{E} \sum_y d_{yz}^2}{\mathbb{E} \sum_y (d_{yz} - \tilde{d}_{yz})^2} \quad (5)$$

(4) where

$$\begin{aligned} d_{yz} - \tilde{d}_{yz} &= \sum_{k, i_k} \epsilon_{i_k} \prod_{\substack{j=1 \\ j \neq k}}^z (f_{i_j} + \epsilon_{i_j}) S_{2^z y - \sum_{k=1}^z 2^{k-1} i_k} \\ &= \sum_{k=1}^z \prod_{\substack{j \in S_k^z \\ 1 \leq i_j \leq N}} \epsilon_{i_j} \prod_{\substack{j' \in \{1: z - S_k^z\} \\ 1 \leq i_{j'} \leq N}} f_{i_{j'}} S_{2^z y - \sum_{k=1}^z 2^{k-1} i_k} \end{aligned} \quad (6)$$

Assuming that $E[S_i S_j] = \delta_{ij}$, we can approximate the numerator as

$$\begin{aligned} \mathbb{E}(d_{yz}^2) &= 1 - \sum_{\{i_k | k=1 \dots z\}} 2 f_{-\sum_{k=2}^z 2^{k-1} i_k} \prod_{k=2}^{z-1} f_{i_k} \\ &+ \sum_{\{i_k, i'_k | k=1 \dots z\}} f_{i_0 + \sum_{k=2}^z 2^{k-1} (i_k - i'_k)} \prod_{k=1}^z f_{i_k} \prod_{k=2}^z f_{i'_k} \end{aligned} \quad (7)$$

Note that the indices $\{i_1, i_k, i'_k, k = 2 \dots z\}$ are many low pass filters which are strictly positive, and the gain can always be adjusted). Under these assumptions, and assuming no exclusions due to truncation, it is easy to show that bounds for the numerator, $\mathbb{E}(d_{yz}^2)$, are

$$\max(0, 1 - 2 \max_k f_k + \min_k f_k) \leq \mathbb{E}(d_{yz}^2) \leq 1 - 2 \min_k f_k + \max_k f_k \quad (8)$$

$$10 \log_{10} \mathbb{E}(\sum_y d_{yz}^2) \leq 10 \log_{10} N - 10z \log_{10} 2 + \log_{10}(1 - 2 \min_k f_k + \max_k f_k) \quad (9)$$

$$10 \log_{10} \mathbb{E} \left(\sum_y d_{yz}^2 \right) \geq 10 \log_{10} N - z \log_{10} 2 + \log_{10} (\max_k (0, 1 - 2 \max_k f_k + \min_k f_k)) \quad (10)$$

These inequalities will be true for real data weight.

with a very small margin of error. However, only inequality 9 is likely to carry much Further note that by assumption, $\mathbb{E}[\epsilon_i \epsilon_j] = 10^{-\text{SNR}_I/10} \frac{\|S^2\|}{N} \delta_{ij}$. Therefore, in general,

$$\begin{aligned} 10 \log_{10} \bar{\mathcal{N}} &= \mathbb{E} \sum_y (d_{yz} - \tilde{d}_{yz})^2 \\ &= 10 \log_{10} \sum_j (10^{-j(\text{SNR}_I/10)} \frac{E \|S\|^{2j} [2j]!}{N^j 2^j j!} \mathbf{f}_j) \\ &= -\text{SNR}_I - 10 \log_{10} \mathbf{f}_1 + 10 \log_{10} \left[1 + \sum_{j=1}^K 10^{-j(\text{SNR}_I/10)} \frac{[2(j+1)]! \mathbf{f}_j}{2^{j+1} j! \mathbf{f}_1} \right] \end{aligned} \quad (11)$$

$$\cong -\text{SNR}_I - 10 \log_{10} \mathbf{f}_1 \quad (12)$$

where \mathbf{f}_j is a factor that depends only on the filter coefficients f_j under the above assumptions. Equation 11 follows since $\|S\|^2 = N$, as S is assumed to be white noise. The expansion in equation 11 involves terms weighting the higher Gaussian moments involving the perturbation of f in equation 4.

The calculation of the expectation of the

denominator in equation 5 is quite complicated. However, if we further assume that the input SNR is sufficiently small so that the summand in equation 11 is negligible, we only need to calculate \mathbf{f}_1 . Under these assumptions, moments higher than the second in the matching random variable need not be considered, and

$$\begin{aligned} \mathbf{f}_1 &= \sum_y \mathbb{E} \left[\sum_{\{i_j, i'_{j'}, j, j'=1 \dots z\}} \epsilon_{i_j} \epsilon_{i'_{j'}} \prod_{\substack{k, k'=1 \\ k \neq j \\ k' \neq j'}}^z f_{i_{k'}} f_{i'_{k'}} S_{2^z y - \sum_{k=1}^z 2^{k-1} i_k} S_{2^z y - \sum_{k=1}^z 2^{k-1} i'_{k'}} \right] \\ &= \sum_y \left[\sum_{\{i_j, i'_{j'}, j, j'=1 \dots z\}} \delta(i_j - i'_{j'}) \prod_{\substack{k, k'=1 \\ k \neq j \\ k' \neq j'}}^z f_{i_{k'}} f_{i'_{k'}} \delta(\sum_{k=1}^z 2^{k-1} \delta(i_k - i'_{k'})) \right] \\ &= \frac{N}{2^z} \left[\sum_{\{i_j, i'_{j'}, j, j'=1 \dots z\}} \prod_{\substack{k, k'=1 \\ k \neq j \\ k' \neq j'}}^z f_{i_{k'}} f_{i'_{k'}} \delta(\sum_{k=1}^z 2^{k-1} (i_k - i'_{k'})) \delta(i_j - i'_{j'}) \right] \end{aligned} \quad (13)$$

Equation 13 shows that we have all products of all filter coefficients of length $2z$, but two

are excluded. The excluded ones represent the same scale, and are required to be evaluated at the same index. One remaining coefficient is fixed so that both products of coefficients would contribute to the same detail coefficient. To obtain a bound on this sum

of products, we again assume $\sum_j f_j = 1$. In this case,

$$N^2 z / 2^z \min |f| \leq f_1 \leq N^2 z / 2^z \max f$$

and thus,

$$10 \log_{10} \frac{\sum_y d_{yz}^2}{\mathbb{E} \sum_y (d_{yz} - d_{yz}^p)^2} \leq \text{SNR}_I - 10 \log_{10} N + 10 \log_{10} (1 - 2 \min_k f_k + \max_k f_k) - 10 \log_{10} z - \log \min f$$

(14)

when there is no truncation due to coefficients being out of bounds.

The two bounds (equations 9 and 14) on the SNR posit that for each scale, the SNR will decrease linearly with slope one as a function of the input SNR, and will fall off logarithmically per octave. That is, the actual data degrades at a rate of -1.6 decibels per octave, and a linear fit to $10 \log_{10} z$ would yield a reduction of 0.92dB per octave. Additional non-linear factors and other factors not treated in this simple model are operative in the numerical simulations.

NUMERICAL ANALYSIS

To study truncation errors numerically, we generated a unit-power white noise time series ($n = 2^{14}$) (our results are qualitatively the same for different classes of time series), downsampled the time series by a factor of two using a FIR low pass filter of order

$N = 30$ (downsampling filter FIR1 [1]), and chose the difference of the low pass filter from the identity as our detail coefficients. Coefficients used in a typical estimation procedure depend on two integers: m , the number of times the data is downsampled prior to estimation, and k , the number of octaves used in the estimation. We used $m = 7$ and $k = 6$ in numerical analyses of the overall process for definiteness. The SNR for each scale was also studied individually. For error analysis, we perturbed the downsampling filter while systematically varying the standard deviation, λ_i , of the noise ϵ_i (equation 4) so that the input signal-to-noise ratio (SNR_I) ranged from 380 to 0 decibels (dB). For each input SNR, we chose 100 white noise time series, perturbed the downsampling filter, and investigated the effect of the input SNR for each time series to the output SNR of the filter given by

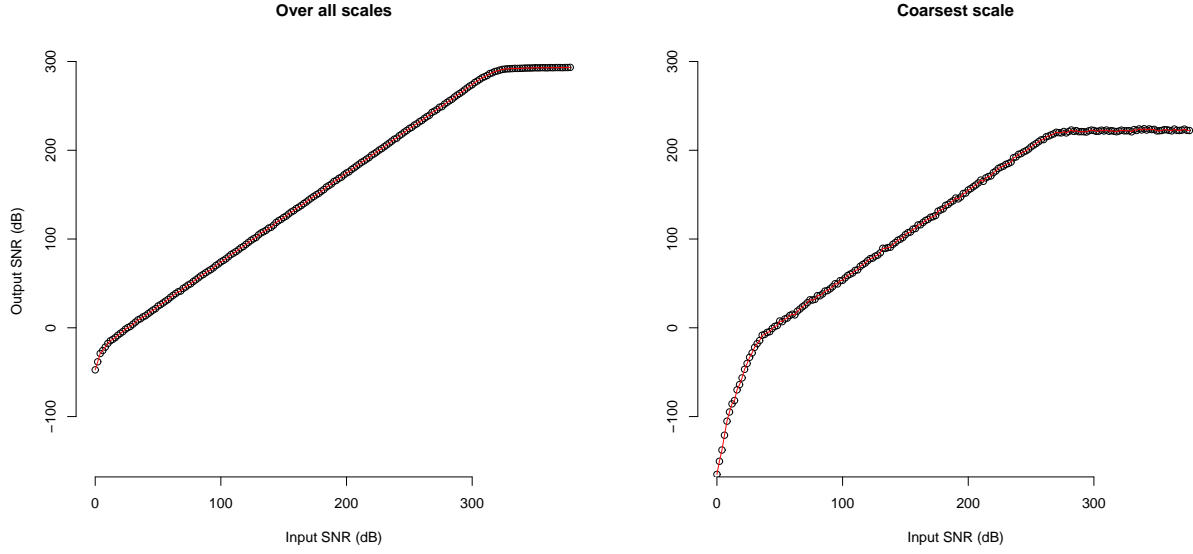


FIG. 1. The relation between input and output SNR for the perturbed wavelet filter over all wavelet scales ($2^2 - 2^9$, left panel) and for the coarse scale (2^2 , right panel) within double machine precision.

$$\mathcal{O}(\text{SNR}_O) = 10 \log_{10} \frac{\sum_{ij} d_{ij}^2}{\sum_{ij} (\tilde{d}_{ij} - d_{ij})^2} \quad (15)$$

where d_{ij} is the i^{th} detail coefficient obtained at level j by subtracting the low pass filtered value from the coefficient prior to downsampling, and \tilde{d}_{ij} is the same coefficient obtained from the perturbed filter (cf. equation 5).

The SNR of the output (SNR_O) was approximately linearly related to that of the input (SNR_I) (Figure 1, left panel) with the equation

$$\mathcal{O}(\text{SNR}_O) = -25.8 + 0.994 \times \mathcal{O}(\text{SNR}_I) \quad (16)$$

This relation shows that on average, a 10^{-4} error in the estimated filter coefficients results in a 10^{-3} error in the output. It should

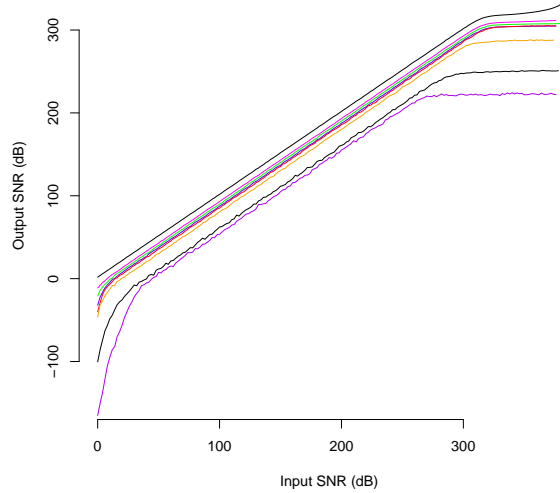


FIG. 2. Degradation of output SNR as a function of scale. Progressively lower traces depict the relation for progressively coarser scales, from 2^{13} to 2^2 .

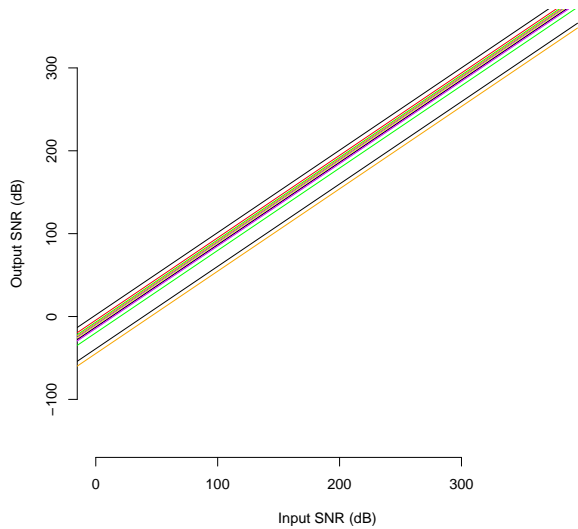


FIG. 3. The linear relation between input and output SNR for scales 2^2 to 2^{13} in the 50 – 270 dB input SNR range. Lower traces show the relation for coarser scales.

be noted, however, that this is a conservative estimate for the overall output error. In fact, stringent constraints may be necessary to attain sufficient accuracy at coarser scales because of the degradation of output SNR (Figure 2).

Complete relation between $\mathcal{O}(\text{SNR}_O)$ and $\mathcal{O}(\text{SNR}_I)$ for the range of double machine precision (Figure 1, right panel) shows that between approximately 50 to 270 dB input SNR, output SNR is approximately a linear function (equation 16). In contrast, output SNR is flat when the input SNR is greater than 270 dB (where less than one bit of noise is added), and accelerates to $-\infty$ from 0 dB

TABLE I. Growing intercept of the relation between input and output SNR with coarser scales. See also Figure 3.

Scale (2^n)	Intercept
2	-44.78
3	-39.86
4	-25.23
5	-14.59
6	-11.26
7	-9.87
8	-7.60
9	-4.61

when the input SNR is below 50 dB. Moreover, although the output SNR is linearly related to that of the input with a slope of 1 for all scales ranging from $k = 2^2$ to $k = 2^9$ within the range of 50 – 270 dB input SNR (Figure 3; cf. equation 16), the intercept of this relation degrades at coarser scales (Table I). Therefore, coarser scales are disproportionately affected. These results highlight that truncation errors in wavelet coefficients can quickly reach to unacceptable levels. For example, suppose that one requires the noise in the output of the coarse scale wavelet coefficients to be less than 1% of the output power. In this case, the noise power in the filter coefficients must be 100 dB, or the filter coefficients must produce errors less than approximately 5×10^{-5} .

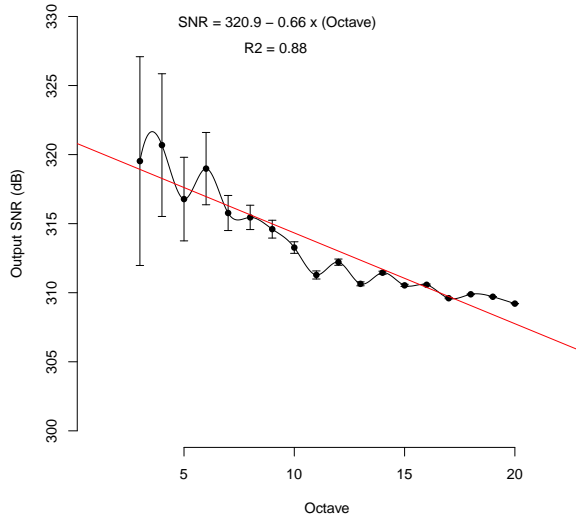


FIG. 4. An error analysis for the FFT of size 2^8 . The output SNR is over 300 dB, and the reduction in precision is about 0.85 for each additional octave. Vertical bars show 95% confidence intervals.

To put this error amplification in perspective, consider simple filtering via Fast Fourier Transform (FFT) of up to size 2^{20} . In this case, SNR of the signal degrades only by about 20 dB from its maximal value of 306 – 325 dB (Figure 4). This relatively small loss indicates that in practice, memory would be exhausted before the truncation error becomes a problem. In contrast, the sub-band coding scheme reported here (thus, most wavelet filters) require a significant length convolution ($N = 30$) at each stage of the calculation. Thus, whereas the FFT requires $\log_2 N$ multiplications for each

of the N elements inverted, wavelet filtering and sub-band coding schemes can require $30^{\log_2 N}$ calculations. This difference can be significant at moderate input SNRs since repeated truncation errors may result in a catastrophic loss of precision.

CONCLUSION

Our results show that truncation error in the wavelet filter coefficients due to quantization and round-off may amplify due to the the multiplicative propagation, and can reach substantial levels. In cases where the relative error in the wavelet coefficients is too large, the wavelet tree (k) grows too deep, or the wavelet filter (N) is too long, this propagation of truncation errors may render the results unreliable especially at coarser scales. Note that although the sub-band analysis presented here is not identical to downsampling using Coiflets or Daubechies filters [2], small errors in the coefficients will still propagate in the latter case because the propagation depends strongly on the length of the filter applied prior to the downsampling. We expect errors with relatively large D15 ($N = 30$ coefficient Daubechies) filter to be comparable to the case presented here.

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